# ON OSCILLATORY CONVECTIVE INSTABILITY OF A CONDUCTING FLUID IN A MAGNETIC FIELD

## (O KOLEBATEL'NOI KONVEKTIVNOI NEUSTOICHIVOSTI PROVODIASHCHEI ZHIDKOSTI V MAGNITNOM POLE)

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M.I. SHLIOMIS
(Perm')

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General studies of the influence of a magnetic field on the convective instability of a conducting fluid shows [1] that all perturbations in the fluid equilibrium develop monotonously; it occurs in weak fields in any case. Oscillatory perturbations , however, were not revealed at all. Nevertheless, as is obvious from a series of simple examples [2 to 4] the oscillatory instability is possible.

The purpose of the present paper is to clarify in a general form the conditions on the arising of oscillatory instability. Such an investigation is necessary, since in all three cited above papers (determination of the beginning of convection in plane horizontal [2] and vertical [3] layers and in a cubic cavity [4]) the oscillatory solutions were obtained only for special boundary conditions on the boundaries of the fluid.

The method adopted here was first used by Landau and Lifshits [5] in studying the intersection of electron terms of molecules.

**1.** In the gravity field

$$\mathbf{g} = -\boldsymbol{\beta} g, \qquad \boldsymbol{\beta}^2 = 1 \tag{1.1}$$

and the external magnetic field

$$\mathbf{H} = \boldsymbol{\gamma} H, \qquad \boldsymbol{\gamma}^2 = 1 \tag{1.2}$$

a conducting fluid which occupies a cavity of an arbitrary shape is heated from below in a manner that a constant temperature gradient is maintained while the fluid is in equilibrium.

$$\nabla T_0 = -\boldsymbol{\beta} A \tag{1.3}$$

Assuming small perturbations in the fluid equilibrium $\mathbf{u}$ , in the temperature $T$ and in the internal magnetic field $\mathbf{h}$ , are proportional to $e^{-\lambda t}$, we obtain from the usual equations of magnetohydrodynamics the following equations for perturbations:

$$-\lambda\mathbf{u} = \nabla^2\mathbf{u} + \Lambda(\gamma\nabla)\mathbf{h} + \Gamma\beta T - \nabla f, \qquad \operatorname{div}\mathbf{u} = 0, \qquad \operatorname{div}\mathbf{h} = 0$$
$$-\lambda\mathbf{h} = a\nabla^2\mathbf{h} + \Lambda(\gamma\nabla)\mathbf{u}, \qquad -\lambda T = b\nabla^2 T + \Gamma\beta\mathbf{u} \tag{1.4}$$

All the quantities here are dimensionless. As unit quantities, we have chosen: length $l$ (characteristic dimension of the cavity), time $l^2/\nu$, velocity $\nu/l$, temperature $\nu l^{-1}(A/\alpha g)^{1/2}$, magnetic field $\nu l^{-1}(4\pi\rho)^{1/2}$. In (1.4) appear the following dimensionless parameters:

$$\Lambda = \frac{Hl}{\nu(4\pi\rho)^{1/2}}, \qquad \Gamma = \frac{l^2}{\nu}(\alpha g A)^{1/2}, \qquad a = \frac{c^2}{4\pi\sigma\nu}, \qquad b = \frac{\chi}{\nu}$$

The number $\Lambda$ determines the ratio of the magnetic energy density $H^2/8\pi$ to the kinetic energy density of the fluid $\rho u^2/2 \sim \rho\nu^2/2l^2$. It does not contain the electric conductivity $\sigma$. The quantity $\Lambda/\sqrt{a}$ is the Hartmann number, and $\Gamma^2$ is the Grasshof number.

At the boundary of the cavity, cut out of an infinite hard conducting solid, the velocity of the fluid vanishes, while the magnetic field, the temperature, the normal component of the heat flux and the tangential component of the electric field are all continuous. The boundary conditions are thus given as

$$\mathbf{u} = 0,\ T = T^\circ, \qquad \mathbf{h} = \mathbf{h}^\circ \quad \text{at the boundary of the cavity}$$
$$\mathbf{n}(\varkappa\nabla T) = \mathbf{n}(\varkappa^\circ\nabla T^\circ), \qquad \frac{\mathbf{n}\times\operatorname{rot}\mathbf{h}}{\sigma} = \frac{\mathbf{n}\times\operatorname{rot}\mathbf{h}^\circ}{\sigma^\circ} \tag{1.5}$$
$$T^\circ = 0, \qquad \mathbf{h}^\circ = 0 \quad \text{at infinity}$$

Here and in what follows, the small superscript circle denotes the values of quantities in the solid ($\varkappa$ is the thermal conductivity). These conditions will allow the use of Gauss's theorem to the entire space for integration; the integrals on the surface of the cavity always vanish.

In this paper, we investigate the dependence of the spectrum of the decrements $\lambda$ on the external field $\Lambda$ and the temperature gradient $\Gamma$. The perturbations are monotonous if $\operatorname{Im}\lambda = 0$ and they damp if $\operatorname{Re}\lambda > 0$.

**2.** Equations (1.4) may be written compactly if we introduce the 7-vector $\psi$ and the operators

$$\psi = \begin{bmatrix}\mathbf{u}\\ \mathbf{h}\\ T\end{bmatrix}, \qquad \nabla f = \begin{bmatrix}\nabla f\\ 0\\ 0\end{bmatrix}, \qquad s_1 = \begin{bmatrix}1&0&0\\0&a&0\\0&0&b\end{bmatrix}, \qquad s_2 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$$
$$s_3 = \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \qquad s_4 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \tag{2.1}$$

System (1.4) may be written

$$\lambda\psi = L\psi \equiv \nabla f - s_1\nabla^2\psi - \Lambda s_2(\gamma\nabla)\psi - \Gamma\beta s_3\psi, \qquad s_4\nabla\psi = 0 \tag{2.2}$$

The operator $L$ is not self-conjugate, so that its eigenvalues may be complex, and its eigenfunctions may not be mutually orthogonal. They are orthogonal to the eigenfunctions $\varphi$ of the operator $L^+$, the Hermitian conjugate of $L$. For $\varphi$ we have Equation

$$\lambda^*\varphi = L^+\varphi \equiv \nabla q - s_1\nabla^2\varphi + \Lambda s_2(\gamma\hat{\nabla})\varphi - \Gamma\beta s_3\varphi, \qquad s_4\nabla\varphi = 0 \quad (2.3)$$

i.e.

$$L^+(\Lambda, \Gamma) = L(-\Lambda, \Gamma) \quad (2.4)$$

Such "weak" non-Hermitian property permits to express $\psi$ and $\varphi$ by the very same functions (cf 2.1):

$$\varphi^* = \begin{bmatrix} \mathbf{u} \\ -\mathbf{h} \\ T \end{bmatrix} \quad (2.5)$$

The vectors $\{\psi_\alpha\}$ are orthogonal to the vectors $\{\varphi_\alpha\}$ in the following sense:

$$(\varphi_\alpha\cdot\psi_\beta) \equiv \int \{\mathbf{u}_\alpha\mathbf{u}_\beta + T_\alpha T_\beta - \mathbf{h}_\alpha\mathbf{h}_\beta\}\, dV = 0 \qquad (\lambda_\alpha \neq \lambda_\beta)$$
$$(\varphi_\alpha\cdot\psi_\alpha) \equiv \int \{\mathbf{u}_\alpha{}^2 + T_\alpha{}^2 - \mathbf{h}_\alpha{}^2\}\, dV = \text{const} \quad (2.6)$$

When $\Lambda \to 0$, the operator $L$ analytically approaches a Hermitian operator, i.e.

$$L^+(0, \Gamma) = L(0, \Gamma) \quad (2.7)$$

so that in the absence of an external magnetic field, oscillatory perturbations do not exist. In [1] it was shown that shuch perturbations also do not occur in weak magnetic fields, i.e. for small values of $\Lambda$ all perturbations are monotone. There exist two types of monotonous perturbations [6]

$$\lambda_{1\alpha}, \psi_{1\alpha} = \begin{bmatrix} \mathbf{u}_{1\alpha} \\ \mathbf{h}_{1\alpha} \\ T_{1\alpha} \end{bmatrix}, \qquad \lambda_{2\alpha}, \psi_{2\alpha} = \begin{bmatrix} \mathbf{u}_{2\alpha} \\ \mathbf{h}_{2\alpha} \\ T_{2\alpha} \end{bmatrix} \qquad (\alpha = 0, 1, 2, \ldots) \quad (2.8)$$

In the solutions of the first type

$$\int \mathbf{h}_{1\alpha}^2\, dV > \int (\mathbf{u}_{1\alpha}^2 + T_{1\alpha}^2)\, dV$$

These solutions may naturally be called "magnetic": as $\Lambda \to 0$, the velocity and temperature vanish in them and only the magnetic field remains and satisfies the Maxwell equations

$$-\lambda_{1\alpha}\mathbf{h}_{1\alpha} = a\nabla^2\mathbf{h}_{1\alpha}, \qquad \operatorname{div}\mathbf{h}_{1\alpha} = 0 \quad (2.9)$$

In the other perturbations, called "hydrodynamic",

$$\int (\mathbf{u}_{2\alpha}^2 + T_{2\alpha}^2)\, dV > \int \mathbf{h}_{2\alpha}^2\, dV$$

In these solutions, as $\Lambda \to 0$, the magnetic field vanishes, and they pass over continuously to the solutions of ordinary convection equations without magnetic field

$$-\lambda_{2\alpha}\mathbf{u}_{2\alpha} = \nabla^2\mathbf{u}_{2\alpha} + \Gamma\beta T_{2\alpha} - \nabla f_{2\alpha}$$
$$-\lambda_{2\alpha}T_{2\alpha} = b\nabla^2 T_{2\alpha} + \Gamma\beta\mathbf{u}_{2\alpha}, \qquad \operatorname{div}\mathbf{u}_{2\alpha} = 0 \quad (2.10)$$

The criterion determining the type of the perturbation is the sign of the normalizing integral

$$(\varphi_{m\alpha}\cdot\psi_{n\beta}) \equiv \int \{\mathbf{u}_{m\alpha}\mathbf{u}_{n\beta} + T_{m\alpha}T_{n\beta} - \mathbf{h}_{m\alpha}\mathbf{h}_{n\beta}\}\, dV = (-)^n\,\delta_{mn}\delta_{\alpha\beta}$$
$$(m,\ n = 1, 2;\quad \alpha,\ \beta = 0, 1, 2 \ldots) \qquad (2.11)$$

Oscillatory perturbations appear for values of $\Lambda$ greater than some critical $\Lambda_*(\Gamma)$. It is essential that they appear in pairs. In fact, from the real character of the operator $L$ follows that if $\{\lambda,\psi\}$ is some complex solution of Equation (2.2), then $\{\lambda^*,\psi^*\}$ is also a solution of this equation. The decrements of oscillatory perturbations $\psi$ and $\psi^*$ at the point $\Lambda_*$ coincide: $\lambda = \lambda^* \equiv \lambda^\circ$, and for $\Lambda < \Lambda_*$ the perturbations themselves become two monotonous perturbations.

Thus, the necessity to study the confluence in the spectrum of the decrements $\lambda(\Lambda,\Gamma)$ arises. This study will be carried out according to the method described in [5].

**3.** Assume that at the point $(\Lambda_0,\Gamma_0)$, the two real decrements $\lambda_{m\alpha}$ and $\lambda_{n\beta}$ have near values. We shall attempt to make $\lambda_{m\alpha} = \lambda_{n\beta}$, by varying the parameters by $\Delta\Lambda$ and $\Delta\Gamma$; we have

$$L(\Lambda,\Gamma) = L(\Lambda_0,\Gamma_0) + \frac{\partial L}{\partial\Lambda}\Delta\Lambda + \frac{\partial L}{\partial\Gamma}\Delta\Gamma \equiv L_0 + \Pi \qquad (3.1)$$

Considering $\Pi$ as a pertutbation on the operator $L_0$, we determine the eigenfunctions and eigenvalues of Equation

$$\lambda\psi = (L_0 + \Pi)\psi \qquad (3.2)$$

by means of the method of the perturbation theory. The eigenfunctions of the "unperturbed" operator $L_0$ satisfy the equations

$$\lambda_{m\alpha}\psi_{m\alpha} = L_0\psi_{m\alpha}, \qquad \lambda_{n\beta}\psi_{n\beta} = L_0\psi_{n\beta} \qquad (3.3)$$

As a first approximation to the eigenfunctions at the point

$$(\Lambda_0 + \Delta\Lambda,\ \Gamma_0 + \Delta\Gamma)$$

we use a linear combination of the type

$$\psi = c_{m\alpha}\psi_{m\alpha} + c_{n\beta}\psi_{n\beta} \qquad (3.4)$$

Substituting (3.4) into (3.2), we get

$$c_{m\alpha}(\lambda - \lambda_{m\alpha} - \Pi)\psi_{m\alpha} + c_{n\beta}(\lambda - \lambda_{n\beta} - \Pi)\psi_{n\beta} = 0 \qquad (3.5)$$

In the problem considered two cases may occur. Perturbations $\psi_{m\alpha}$ and $\psi_{n\beta}$, whose decrements have the close values at the point $(\Lambda_0,\Gamma_0)$, may belong (a) to one type $(n = m)$, or (b) to different types $(n \neq m)$.

To begin, let us examine the first case, that id both $\psi_{n\alpha}$ and $\psi_{n\beta}$ are either "magnetic" $(n = 1)$, or "hydrodynamic" $(n=2)$ perturbations. By taking the inner product of (3.5) with $\varphi_\alpha$ and $\varphi_\beta$ in turn (dropping the index $n$), we get two algebraic equations, which are solvable if

$$\begin{vmatrix} (-)^n(\lambda_\alpha - \lambda) + \Pi_{\alpha\alpha} & \Pi_{\alpha\beta} \\ \Pi_{\beta\alpha} & (-)^n(\lambda_\beta - \lambda) + \Pi_{\beta\beta} \end{vmatrix} = 0 \qquad (3.6)$$

Here

$$\Pi_{\alpha\beta} \equiv (\varphi_\alpha \cdot \Pi\psi_\beta) =$$

$$= - \Delta\Lambda \int \{\mathbf{u}_\alpha (\gamma\nabla)\, \mathbf{h}_\beta + \mathbf{u}_\beta (\gamma\nabla)\, \mathbf{h}_\alpha\}\, dV - \Delta\Gamma \int \{T_\alpha \beta \mathbf{u}_\beta + T_\beta \beta \mathbf{u}_\alpha\}\, dV \qquad (3.7)$$

From (3.7) it is seen that

$$\Pi_{\alpha\beta} = \Pi_{\beta\alpha} \qquad (3.8)$$

(The matrix $\Pi_{\alpha\beta}$ is determined in the mixed basis $\{\varphi_\alpha; \psi_\alpha\}$, and therefore (3.8) does not mean, of course, that $\Pi$ is Hermitian.)

By expanding the determinant (3.6), we find

$$\lambda = {}^1/_2\, [\lambda_\alpha + \lambda_\beta + (-)^n (\Pi_{\alpha\alpha} + \Pi_{\beta\beta})] \pm \qquad (3.9)$$

$$\pm \sqrt{{}^1/_4\, [\lambda_\alpha - \lambda_\beta + (-)^n (\Pi_{\alpha\alpha} - \Pi_{\beta\beta})]^2 + \Pi_{\alpha\beta}^2}$$

For the confluence of the decrements the expression under the radical must be made zero. Since this appears as the sum of two squares, the conditions for confluence consist of the two equations

$$\lambda_\alpha - \lambda_\beta + (-)^n (\Pi_{\alpha\alpha} - \Pi_{\beta\beta}) = 0, \qquad \Pi_{\alpha\beta} = 0 \qquad (3.10)$$

With two arbitrary parameters, $\Delta\Lambda$ and $\Delta\Gamma$, which determine the perturbation $\Pi$, these equations can always be satisfied. Consequently the decrements of any two perturbations of the same type may intersect. Nothing of interest, however, arises from this, since such a confluence has no relation to an oscillatory perturbation. The decrements determined by Formula (3.9) separate after the confluence and remain real.

If $\psi_\alpha$ and $\psi_\beta$ possess different symmetry, then [5] $\Pi_{\alpha\beta} \equiv 0$, and from the two conditions for confluence (3.10) only one remains. Therefore the intersection of the surfaces $\lambda_\alpha(\Lambda, \Gamma)$ and $\lambda_\beta(\Lambda, \Gamma)$ occurs along a line for perturbations of opposite symmetry and at a point for those of the same symmetry.

4. It remains to consider the case when at the point $(\Lambda_0, \Gamma_0)$ the close values $\lambda_{1\alpha}$ and $\lambda_{2\beta}$ are decrements of two perturbations of different type. In the following the second subscript will be omitted. Multiplying (3.5) by $\varphi_1$ and $\varphi_2$ we get a system of two equations, the condition of solvability of which is

$$\begin{vmatrix} \lambda_1 - \lambda - \Pi_{11} & \Pi_{12} \\ -\Pi_{21} & \lambda_2 - \lambda + \Pi_{22} \end{vmatrix} = 0 \qquad (4.1)$$

Here $\Pi_{\alpha\beta}$ is determined by (3.7), so that $\Pi_{21} = \Pi_{12}$. From this by expanding the determinant we get

$$\lambda = {}^1/_2\, (\lambda_1 + \lambda_2 - \Pi_{11} + \Pi_{22}) \pm \sqrt{{}^1/_4\, (\lambda_1 - \lambda_2 - \Pi_{11} - \Pi_{22})^2 - \Pi_{12}^2} \qquad (4.2)$$

If $\psi_1$ and $\psi_2$ have different symmetries, then $\Pi_{12} \equiv 0$, and the expression under the radical in (4.2) is always nonnegative; that is, a complex $\lambda$ is impossible. The confluence of the decrements, which is possible here, was discussed in Section 3.

The most interesting case is that where $\psi_1$ and $\psi_2$ have the same symmetry. Then $\Pi_{12} \not\equiv 0$, so that now under the radical is not a sum, as in (3.9), but a difference of two squares. With the proper choices of $\Delta\Lambda$ and $\Delta\Gamma$ this difference can be positive, negative, or equal to zero. Negative values of the difference correspond to two complex-conjugate decrements, that is, two oscillatory perturbations with frequencies $\pm \operatorname{Im}\lambda$. With a change of sign of the expression under the radical this pair of oscillatory perturbations goes over into two monotonous perturbations: one "magnetic" and one "hydrodynamic". The condition for confluence of the decrements consists of Equation

$$^1/_4\,(\lambda_1 - \lambda_2 - \Pi_{11} - \Pi_{22})^2 - \Pi_{12}{}^2 = 0 \tag{4.3}$$

that is, the surfaces $\lambda_1(\Lambda, \Gamma)$ and $\lambda_2(\Lambda, \Gamma)$ intersect along a line. Let us determine the rule according which these decrements change in the neighborhood of this line. If the point $\Lambda_* = \Lambda_0 + \Delta\Lambda$, $\Gamma_* = \Gamma_0 + \Delta\Gamma$ lies on the line of intersection of the decrements, then in the neighborhood of this point

$$\Lambda = \Lambda_* + \xi \equiv \Lambda_0 + \Delta\Lambda + \xi, \qquad \Gamma = \Gamma_* + \eta \equiv \Gamma_0 + \Delta\Gamma + \eta \tag{4.4}$$

Taking into account that $\lambda_1(\Lambda_*, \Gamma_*) = \lambda_2(\Lambda_*, \Gamma_*) \equiv \lambda^\circ$, we get from (4.2) for the values of the parameters in (4.4)

$$\lambda = \lambda^\circ \pm \sqrt{B\xi + D\eta} \qquad (B, D = \text{const}) \tag{4.5}$$

Thus, near $(\Lambda_*, \Gamma_*)$ the frequency of oscillations is

$$\sqrt{B\,(\Lambda - \Lambda_*) + D\,(\Gamma - \Gamma_*)} \tag{4.6}$$

Thus the reason for the appearance of oscillatory perturbations is the confluence of the decrements of monotonous perturbations of different type, but the same symmetry. For this case the upper portion of Fig.1 shows the relief sections of the function $\lambda(\Lambda^2, \Gamma^2)$ for four different planes $\Gamma^2$ = const. To the right of the point of confluence of the "magnetic" and "hydrodynamic" decrements appear two complex conjugate $\lambda$. On the Fig.1 their real parts are designated.

**5.** The boundary value problem (1.4) with $\Lambda = 0$ breaks up into two problems (2.9) and (2.10). Problem (2.10) has been investigated by Sorokin [7]. By the use of a variational technique he showed that with the growth of $\Gamma$ all $\lambda_{2\alpha}$ are decreased. For $\Gamma = \Gamma_2$ (see the upper part of Fig.1) the decrement $\lambda_2$ becomes zero. For $\Gamma > \Gamma_2$ the monotonous perturbation $\psi_2$ would be strengthened, leading to instability.

The eigen numbers of $\lambda_{1\alpha}$ of the other problem (2.9) do not depend on $\Gamma$ at all. Therefore $\lambda_1(0, \Gamma^2)$ is a common point of all curves corresponding to different values of $\Gamma^2$.

In the presence of a magnetic field ($\Lambda \neq 0$) two forms of convective instability corresponding to monotonous and oscillatory perturbations, are possible. In weak fields, as long as $\Lambda$ is less than some $\Lambda^\circ$, equilibrium is threatened only by monotonous perturbations, and the critical value of $\Gamma$, above which the equilibrium is unstable, increases with increasing $\Lambda$. Oscillatory instability occurs for $\Lambda > \Lambda^\circ$.

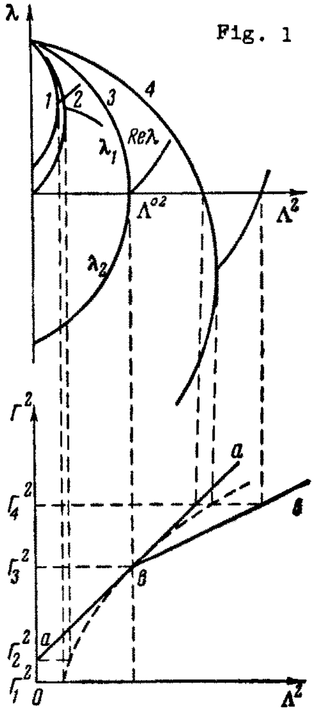

Fig. 1

The lower part of Fig.1 is taken from [3]. The solid lines on it designate the limits of stability for plane vertical layer of fluid, heated from below with a transverse magnetic field. Line *aa* determines the monotonous, while *bb* gives the oscillatory threshold of convection. The region of existence of oscillatory perturbations lies under the dotted curve. This curve projects on the plane $\mathrm{Re}\,\lambda = 0$ just that line along which the surfaces $\lambda_1(\Lambda^2, \Gamma^2)$ and $\lambda_2(\Lambda^2, \Gamma^2)$ intersect.

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